An Intuitive Approach to Normal Operators

Benoit Dherin, Dan Sparks

The question

In this short document I will transcribe and slightly modify an e-mail sent to me by Benoit Dherin, which explains how one would be lead to the definition of a normal operator.

The usual definitions are: the adjoint T^* of T is the unique operator satisfying $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for all v, w. A normal operator is one such that $T^*T = TT^*$.

The question is, where on earth does this definition *come from*?

Consider the following answers: (1) Because complex numbers satisfy $z\bar{z} = \bar{z}z$. (2) Because operators with an orthonormal eigenbasis satisfy $TT^* = T^*T$.

The reason that these answers do not address the question is because they are inherently unnatural. The point is that complex numbers, and operators with orthonormal eigenbases, have *many* properties. Not every one of those properties is worth centering a definition around, so there should be a *reason* to look at normal operators in particular.

Groundwork; discovering self-adjoint operators

Let V be a finite dimensional complex inner product space. The most natural way to think about $\mathcal{L}(V)$ is as a noncommutative ring without unit (or as an analyst might simply say, a ring). The prototype of this type of ring is the set of $n \times n$ matrices, which $\mathcal{L}(V)$ "is," up to a non-canonical isomorphism. (For each basis of V, there is an isomorphism $\mathcal{L}(V) \to M_{n\times n}(\mathbf{C})$. Better, for each orthonormal basis of V, there is an isomorphism $\mathcal{L}(V) \to M_{n\times n}(\mathbf{C})$ which respects the norm nicely (c.f. "operator norm").)

The complex numbers themselves are a wonderful ring, even a field, which have many nice properties. One nice thing about the complex numbers is that they have an involution $* : z \mapsto \bar{z}$. This respects virtually all the structure of $\mathbf{C}$: it is a continuous field automorphism, and its fixed field, i.e. $\{z \in \mathbf{C} : z = \bar{z}\}$ is exactly $\mathbf{R}$. (The Galois group $\mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \{\mathrm{Id}, *\}$.) This involution also has nice properties and is useful for analyzing the complex numbers.

So it would be nice if we could have an involution satisfying as many of the properties of $*$ as possible. The first involutions that people might think of on $\mathcal{L}(V)$ are "take the complex conjugate of all the matrix entries" or "take the transpose matrix." Notice that $\mathbf{C}$ sits canonically inside of $\mathcal{L}(V)$, as the one dimensional subspace $\{z\mathrm{Id} : z \in \mathbf{C}\}$. The transpose is immediately suspect since $(zT)^t = z(T^t)$. We might prefer that our "$*$" map satisfy $(zT)^* = \bar{z}T^*$ if it is really trying to emulate complex conjugation. One may have some luck proving things about the other involution, namely "take the complex conjugate of the matrix entries." Then "self-adjoint" would mean "has real entries," etc.

But if one were unsatisfied with this involution, the next thing to try would be to combine the two: consider actually the conjugate transpose. This is where the theory would really take off.

If T is an operator and $\{e_1, \cdots, e_n\}$ is an orthonormal basis, then the matrix $\mathcal{M}(T)$ has i-th, j-th entry $\langle Te_j, e_i \rangle$. So one way to encode "conjugate transpose" would be to say that

$$\langle T^* e_j, e_i \rangle = \overline{\langle Te_j, e_i \rangle} = \langle e_i, e_j \rangle$$

which, quantified over all v, w instead of just over a basis, gives us our modern day definition of adjoint.

From here, we return to the task of analyzing the ring $\mathcal{L}(V)$. As we said before, it is useful to identify the invariants of the involution, i.e. the real numbers. This leads us to define the self-adjoint operators $T = T^*$, which would play the role of the real numbers in our analogy. This definition would then be wildly successful: we would have no trouble proving

Niave spectral theorem: Every self-adjoint operator admits an orthonormal eigenbasis.

The first explanation

If self adjoint operators are supposed to be analogous to real numbers, and since every complex number $z = a + ib$ for real a, b, and since the complex numbers are such a marvelous ring, perhaps operators of the form $R + iM$ would be interesting. With this definition in hand, we would likely have no trouble proving the complex spectral theorem and that an operator is of the form $T = R + iM$ if and only if $TT^* = T^*T$. So the analogy becomes the rather beautiful

$$\begin{array}{rcl} \text{Complex numbers} & \leftrightarrow & \text{Normal operators} \\ \text{Real numbers} & \leftrightarrow & \text{Self adjoint operators} \end{array}$$

The second explanation will have more details.

The second explanation

Given a theorem (in our case, we imagine that we've proven self adjoint $\Rightarrow \exists$ orthonormal eigenbasis) one naturally seeks to prove a converse, or in failing to do so, either (a) discover a possible weakening of hypotheses or (b) discover a strengthening of the conclusion, so as to have a biconditional theorem.

Let's take a look. Suppose we are given an operator T which has an orthonormal eigenbasis. Then, with respect to this basis, it's matrix is diagonal. If you write this out, and compare it with its conjugate transpose, we see that the converse is false

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \neq \begin{pmatrix} \overline{\lambda_1} & 0 & \cdots & 0 \\ 0 & \overline{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \overline{\lambda_n} \end{pmatrix}$$

unless, of course the λ_i's are real numbers. This quickly leads to a second version of the theorem, which would be

Pre-cursor to the spectral theorem: T is self adjoint if and only if it admits an orthonormal basis of eigenvectors with real eigenvalues.

On the other hand, suppose we want to weaken the hypotheses. In other words, we want a definition of a type of operator which is not quite as restrictive as that of a real number, and we'd like to be able to prove that the operator satisfies this definition by assuming only that it has an orthonormal eigenbasis. Let's write our eigenvalues $\lambda_j = \alpha_j + i\beta_j$, and suppose again that T has an orthonormal eigenbasis:

$$\mathcal{M}(T) = \begin{pmatrix} \alpha_1 + i\beta_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 + i\beta_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha_n + i\beta_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix} + i \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \beta_n \end{pmatrix}$$

So what we're seeing is that every operator with an orthonormal eigenbasis can be written as $R + iM$, where R and M actually *are* self-adjoint. (See also the first explanation, which may play a role in the following insight:) If one then supposes they have an operator of the form $R + iM$ they will have no trouble proving a converse, thereby establishing the complex spectral theorem as we know it now:

Complex pectral theorem: An operator T is of the form $T = R + iM$, for R, M self-adjoint, if and only if T admits an orthonormal basis of eigenvectors. Moreover, $M = 0$ (i.e., T is self-adjoint) if and only if the eigenvalues are real.

So we define a normal operator to be one which is of the form $R + iM$. As we know, once you have a definition, you try to find equivalent definitions! Form here there would be no trouble finding the definition $T^*T = TT^*$.

The third explanation

Professor Dherin's original intuition varied slightly from this. After discovering the naive spectral theorem, and how nice self-adjoint operators are, one might want to look at the analogue of the unit circle $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. (The unit circle is very important in various ways in complex analysis and various branches of mathematics where complex analysis is applied, like number theory (one thing that comes to mind is the proof of Dirichlet's unit theorem).) Now we know that $|z| = 1 \Leftrightarrow z\bar{z} = \bar{z}z = 1$, and the latter condition can actually be phrased in terms of operators. This leads to a nice definition of unitary operators, which are operators which satisfy exactly this condition: $T^*T = TT^* = 1$. (This is equivalent to T carrying some (hence, every) orthonormal basis to an orthonormal basis, or in other words, "the columns of T are an orthonormal basis.") The success of the definition of a unitary operator, and especially realizing how useful the condition $T^*T = TT^*$ is while proving things about unitary operators, one might consider weakening the definition to simply $T^*T = TT^*$ and seeing which theorems are still true, which would consequently lead to our definition of normal operator.

The fourth explanation

A last bit of intuition for the origin of normalcy, one might naively ask whether $T^*T = TT^*$ in general, especially if one has just defined a unitary operator. After finding a counterexample, one might try to characterize which operators have this property. If one simply assumes that $T^*T = TT^*$, one will quickly prove the spectral theorem and have discovered normal operators.